Google Scholar Crossref doi scopus

Impact factor 6.2

Geoscience Journal

ISSN:1000-8527

Indexing:

- Scopus
- Google Scholar
- DOI, Zenodo
- Open Access

www.geoscience.ac

Registered

APPROXIMATE SOLUTION OF FRACTIONAL BACTERIAL GROWTH MODEL USING MAHGOUB ADOMIAN DECOMPOSITION METHOD

Dr. S. Katrin Lydia[1] and Dr. D. Muthuselvam[2]

[1]Faculty-Mathematics, School of Marine Engineering and Technology, Indian Maritime University, Chennai Campus, Tamil Nadu, India.

[2]Assistant Professor, Department of Botany, Bishop Heber College, Affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.

Abstract

This study presents an approximate analytical solution for a fractional-order bacterial growth model by applying the Mahgoub–Adomian Decomposition Method (MADM). This technique combines the Mahgoub Transform with the classical Adomian decomposition approach to treat nonlinear fractional partial differential equations. The model captures the growth behavior of *Bacillus subtilis* colonies expanding on nutrient agar. Fractional derivatives are considered in the Caputo sense to better represent the memory-driven diffusion and reaction processes. Numerical results show that this method is easy to implement and accurate when applied to fractional partial differential.

Introduction

Fractional Calculus is a valuable mathematical extension that generalizes differentiation and integration to non-integer (arbitrary real) orders. This led to the creation of fractional differential equations (FDEs), which are superior instruments for modelling many complex physical phenomena. Currently, FDE theory is a major point of interest for researchers because of its diverse and effective applications across fields, including Bioengineering, Electrochemical processes, Viscoelasticity, Electromagnetic theory, Circuit Theory, and Atmospheric Physics [1-4]

Levine and Jacob [5] studied the pattern formation in bacterial colonies and they delivered the result that the growth and development of bacteria population shows a great variety of geometrical shapes. Adler [6] was the first person who obtained bacterial population waves in the form of concentric circle. Golding et al. [7] examined the growth of bacteria when nutrient supply is limited. Matsushita et. al., [8] showed the growth of bacteria under nutrient rich conditions. Many numerical techniques have been introduced to solve the fractional reaction diffusion model. Adomian decomposition method [9] is one of the powerful methods has been introduced for finding the solution of generalized reaction diffusion model for bacterial colony. Salah and Hassan [10] applied Homotopy analysis transform method to find the numerical solution of fractional reaction diffusion model. Haidong et.al., [11] used Adams-Bashforth method to find the approximate solution of fractional bacterial dependent diseases.

Yeolekar et.al., [12] applied Homotopy Pertubation Transform Method to obtain an analytical solution of fractional order bacterial disease model.

Now, the fractional order reaction diffusion equation is given by [10],

$$\frac{\partial^\alpha b}{\partial t^\alpha} = D_b \frac{\partial}{\partial x}\left(nb \frac{\partial b}{\partial x}\right) + nb,$$

$$\frac{\partial^\beta n}{\partial t^\beta} = D_n \frac{\partial^2 n}{\partial x^2} - nb \qquad (1)$$

with initial condition

$$b(x,0) = b_0(x),$$

$$n(x,0) = n_0(x),$$

where D_b, D_n are the diffusion coefficients describe the bacterial cell movements and nutrient, $b(x,t)$ denotes the density of bacteria projected on plane, $n(x,t)$ the concentration of the nutrient, n_0 is the initial concentration of the nutrient and b_0 is the density of the initial concentration. Eqn. (1) is reaction-diffusion type model which described as the evolution of bacterium *Bacillus subtilis*. This bacterium was grown on the surface of thin agar plates and it can be developed in various types of spatial patterns such as rings, disks and spots. Fig. (1) shows the Bacterium *Bacillus subtilis* and Fig. (2) provides the cross section of the Bacterium *Bacillus subtilis.*

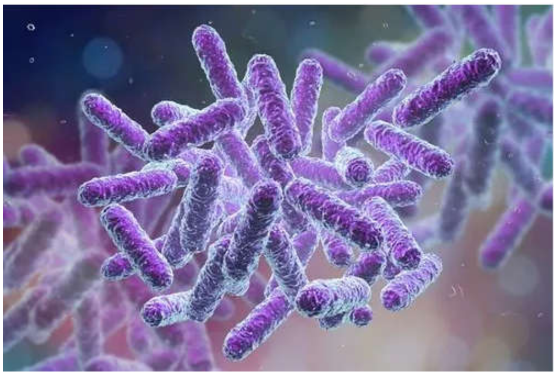

Fig. (1) Bacterium *Bacillus subtilis*

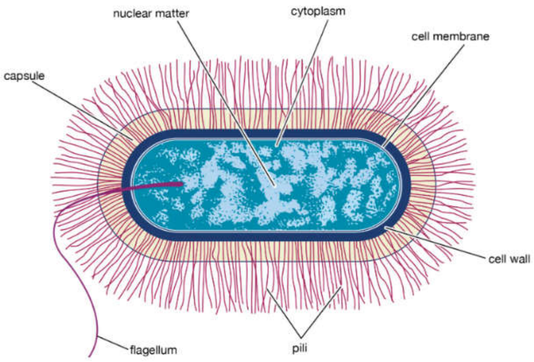

Fig. (2) Cross Section of Bacterium *Bacillus subtilis*

Bacillus subtilis is a common, Gram-positive, rod-shaped bacterium naturally found in soil and vegetation. Christian Gottfried Ehrenberg (a German microscopist) first described this bacterium in 1835. He originally named it *Vibrio subtilis*, likely due to the cells' movement (vibration). Ferdinand Julius Cohn (another German botanist and microbiologist) later reclassified it and gave it the name *Bacillus subtilis* in 1872. Cohn is also credited with discovering that *B. subtilis* can form heat-resistant spores, a finding that helped lead to the development of pasteurization.

The structure of this paper is outlined below. Section 2 establishes the fundamental concepts of Fractional Calculus and the Mahgoub Transform. Section 3 details the construction of the Mahgoub Adomian Decomposition Method (MADM). Section 4 discusses the numerical solutions derived from the proposed mathematical model and Section 5 provides the conclusion.

2. Preliminaries and Notations

This section presents some important definitions of fractional calculus and Mahgoub transform and its properties.

Definition 2.1: A real function $\mathrm{f}(\mathrm{t}), \mathrm{t} > 0$ is said to be in the space $\mathrm{C}_{\mu},\ \mu \in \mathbb{R}$ if there exists a real number $\mathrm{p} > \mu$ such that $\mathrm{f}(\mathrm{t}) = \mathrm{t}^{\mathrm{p}}\mathrm{f}_1(\mathrm{t})$ where $\mathrm{f}_1(\mathrm{t}) \in \mathrm{C}[0,\infty)$ and it is said to be in the space $\mathrm{C}_{\mu}^{\mathrm{n}}$ if and only if $\mathrm{f}^{(\mathrm{n})} \in \mathrm{C}_{\mu},\ \mathrm{n} \in \mathbb{N}$ [13].

Definition 2.2: The partial Riemann-Liouville fractional derivative ${}^{RL}\partial_t^{\alpha}$ of order $\alpha \in R, \alpha > 0$ of function $f(x,t) \in C_{\mu}, \mu \geq -1$ is defined as [14]

$$\begin{aligned} {}^{RL}\partial_t^{\alpha} f(x,t) &= \left(\frac{\partial}{\partial t}\right)^n \left(\partial_t^{\alpha-n} f(x,t)\right) \\ &= \frac{1}{\Gamma(\mathrm{n}-\alpha)}\left(\frac{\partial}{\partial t}\right)^n \int_0^t \frac{f(x,\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \quad t > 0 \end{aligned} \tag{2}$$

where $n-1 < \alpha < n$

Definition 2.3: The partial Caputo fractional derivative ${}^{c}\partial_t^{\alpha}$ of order $\alpha \in R, \alpha > 0$ of function $f(x,t)$ is defined as: [15]

$${}^{c}\partial_t^{\alpha} f(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n f(x,t)}{\partial t^n}, & \alpha = n \end{cases} \tag{3}$$

Definition 2.4: Mahgoub Transform is defined on the set of continuous functions and exponential order. We consider functions in the set A defined by [16]

$$A = \left\{ f(t): |f(t)| < Pe^{\frac{|t|}{\epsilon_i}}\ if\ t \in (-1)^i \times [0,\infty), i = 1,2;\ \epsilon_i > 0 \right\} \tag{4}$$

where ϵ_1, ϵ_2 may be finite or infinite and the constant P must be finite.

Let $f \in A$, then Mahgoub Transform is defined as

$$M[f(t)] = H(u) = u\int_0^{\infty} f(t)e^{-ut}dt,\ \ t \geq 0, \in_1 \leq u \leq \in_2 \tag{5}$$

Theorem 2.5: Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $H(x,u)$ be the Mahgoub Transform of the function $f(x,t)$, then the Mahgoub Transform of partial Caputo fractional derivative of $f(x,t)$ of order α is given by [17]

$$M[{}^{c}\partial_t^{\alpha} f(x,t)] = u^{\alpha}H(x,u) - \sum_{k=0}^{n-1} u^{\alpha-k} f^{(k)}(x,0), n-1 < \alpha \leq n,\ \ n \in \mathbb{N} \tag{6}$$

3. Construction of Maghoub Adomian Decomposition Method in Fractional Bacterial Growth Model

In this section, the procedure for solving fractional bacterial growth model (1) with given initial conditions have been discussed.

Applying Mahgoub Transform on both sides of Eqn. (1) as

$$M\left\{\frac{{}^{c}\partial^{\alpha} b}{\partial t^{\alpha}}\right\} = M\left\{D_b \frac{\partial}{\partial x}\left(nb\frac{\partial b}{\partial x}\right) + nb\right\}$$

$$M\left\{\frac{{}^{c}\partial^{\beta} n}{\partial t^{\beta}}\right\} = M\left\{D_n \frac{\partial^2 n}{\partial x^2} - nb\right\}$$

Using Theorem 2.5, we have

$$u^{\alpha}M\{b(x,t)\} - u^{\alpha}b(x,0) = M\left\{D_b \frac{\partial}{\partial x}\left(nb\frac{\partial b}{\partial x}\right) + nb\right\}$$

$$u^{\beta}M\{n(x,t)\} - u^{\beta}n(x,0) = M\left\{D_n \frac{\partial^2 n}{\partial x^2} - nb\right\} \tag{7}$$

Now, using initial conditions and taking inverse Mahgoub Transform in Eqn. (7)

$$b(x,t) = b(x,0) + M^{-1}\left[\frac{1}{u^{\alpha}} M\left\{D_b \frac{\partial}{\partial x}\left(nb\frac{\partial b}{\partial x}\right) + nb\right\}\right]$$

$$n(x,t) = n(x,0) + M^{-1}\left[\frac{1}{u^{\beta}} M\left\{D_n \frac{\partial^2 n}{\partial x^2} - nb\right\}\right] \tag{8}$$

Assuming that the solutions, $b(x,t),\ n(x,t)$ in the form of infinite series given by

$$b(x,t) = \sum_{m=0}^{\infty} b_m(x,t),\quad n(x,t) = \sum_{m=0}^{\infty} n_m(x,t) \tag{9}$$

and the nonlinear terms involved in the model $nb\frac{\partial b}{\partial x}$ and $n(x,t)b(x,t)$ are decomposed by Adomian polynomials as

$$nb\frac{\partial b}{\partial x}=\sum_{m=0}^{\infty}A_m \tag{10}$$

$$n(x,t)b(x,t)=\sum_{m=0}^{\infty}B_m \tag{11}$$

where A_m, B_m are Adomian polynomials and the first three polynomials are given by

$$A_0=n_0b_0\frac{\partial b_0}{\partial x}$$

$$A_1=(n_0b_1+n_1b_0)\frac{\partial b_0}{\partial x}+n_0b_0\frac{\partial b_1}{\partial x}$$

$$A_2=(2n_1b_1+n_0b_2+n_2b_0)\frac{\partial b_0}{\partial x}+(2n_1b_0+n_0b_1)\frac{\partial b_1}{\partial x}+n_0b_0\frac{\partial b_2}{\partial x}$$

$$\vdots$$

and

$$B_0=n_0b_0,$$

$$B_1=n_0b_1+n_1b_0$$

$$B_2=n_0b_2+n_1b_1+n_2b_0$$

$$\vdots$$

Subject to the initial conditions

$$b(x,0)=e^{-ax}$$

$$n(x,0)=1$$

Using Eqns. (9) - (11) in Eqn. (8) we get

$$b_0=e^{-ax}$$

$$b_1=(2a^2D_be^{-ax}+1)e^{-ax}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$b_2=(18a^4D_b^2e^{-2ax}+6a^2D_be^{-ax}+1)e^{-ax}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-(3a^2D_be^{-ax}+1)e^{-2ax}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$\vdots$$

and

$$n_0=1$$

$$n_1 = -e^{-ax}\frac{t^{\beta}}{\Gamma(\beta+1)}$$

$$n_2 = (e^{-ax} - a^2 D_n)e^{-ax}\frac{t^{2\beta}}{\Gamma(2\beta+1)} - (2a^2 D_b e^{-ax} + 1)e^{-ax}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$\vdots$$

By substituting values of $b_0, b_1, b_2 \ldots$ and $n_0, n_1, n_2 \ldots$ in Eqn. (5) gives the approximate solution of $b(x,t)$ and $n(x,t)$ in series form as

$$b(x,t) = b_0 + b_1 + b_2 \ldots$$

$$n(x,t) = n_0 + n_1 + n_2 \ldots$$

Thus,

$$b(x,t) = e^{-ax} + (2a^2 D_b e^{-ax} + 1)e^{-ax}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \ldots$$

$$n(x,t) = 1 - e^{-ax}\frac{t^{\beta}}{\Gamma(\beta+1)} + (e^{-ax} - a^2 D_n)e^{-ax}\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \cdots$$

4. Numerical Results

The approximate series solutions for $b(x,t)$ and $n(x,t)$ are computed at selected values of x and t for different values of the fractional order α *and* β. Tables 4.1 and 4.2 show the approximate solution of Bacterial growth and concentration of nutrients by MADM in case of $D_b = 0.01, a = 1.2, D_n = 1$ at $t = 0.01$ for different values of α *and* β. Figs. 3 and 4 present the graphical representation of Bacterial growth and concentration of nutrients by MADM for different values of α *and* β.

Table 4.1: Bacterial growth b (x, t) in case $D_b = 0.01, a = 1.2, D_n = 1$ at $t = 0.01$

x	$\alpha = 0.25$; $\beta = 0.5$	$\alpha = 0.4$; $\beta = 0.5$	$\alpha = 0.5$; $\beta = 0.75$	$\alpha = 0.75$; $\beta = 0.95$	$\alpha = 1$; $\beta = 1$
- 0.4	2.319594	1.922237	1.817693	1.674945	1.632940
- 0.2	1.832410	1.516806	1.429373	1.317219	1.284413
0	1.446044	1.195981	1.124077	1.035950	1.010290
0.2	1.140271	0.942486	0.884034	0.814775	0.794682
0.4	0.898641	0.742411	0.695280	0.640843	0.625094

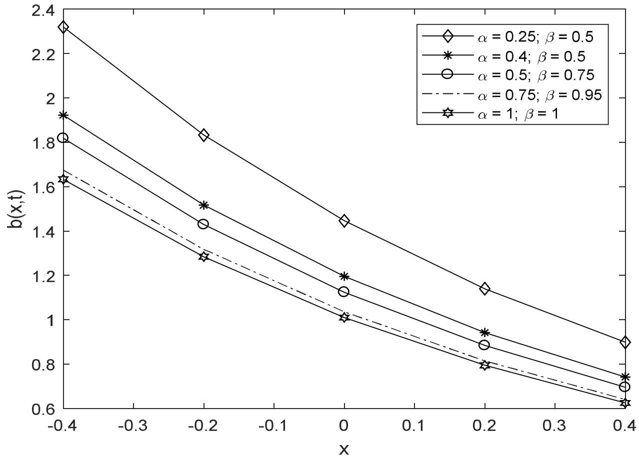

Fig. 3: Numerical solution of Bacterial growth of Eqn. (1)

Table 4.2: Concentration of nutrient n (x, t) in case $D_b = 0.01, a = 1.2, D_n = 1$ at $t = 0.01$

x	$\alpha = 0.25$; $\beta = 0.5$	$\alpha = 0.4$; $\beta = 0.5$	$\alpha = 0.5$; $\beta = 0.75$	$\alpha = 0.75$; $\beta = 0.95$	$\alpha = 1$; $\beta = 1$
-0.4	0.762298	0.792620	0.939888	0.978826	0.983769
-0.2	0.809068	0.832694	0.952420	0.983309	0.987211
0	0.847363	0.865808	0.962390	0.986849	0.989927
0.2	0.878420	0.892842	0.970302	0.989642	0.992068
0.4	0.903426	0.914718	0.976569	0.991844	0.993755

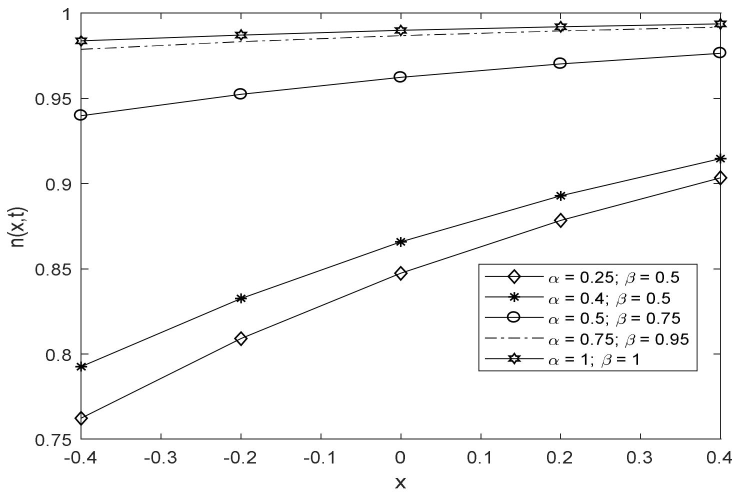

Fig. 4: Numerical solution of Concentration of Nutrients of Eqn. (1)

5. Conclusion

This work investigated a fractional reaction–diffusion model that characterizes the spatial and temporal development of *Bacillus subtilis* colonies. By employing the Mahgoub–Adomian Decomposition Method, approximate analytical solutions are derived without resorting to heavy computational schemes. The method demonstrates simplicity, adaptability, and strong potential for solving nonlinear fractional partial differential equations encountered in biological and physical systems. These findings suggest that MADM is a valuable mathematical tool for modelling real-world processes involving anomalous diffusion and memory effects.

References

[1] Kilbas AA, Srivastava HM and Trujillo JJ., Theory and Applications of Fractional Differential Equations North-Holland Mathematical Studies, 204, Amsterdam, London and New York: Elsevier (North-Holland) Science Publishers; 2006.

[2] Podlubny I, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, 198, San Diego, London, and Toronto: Academic Press; 1999.

[3] Sabatier J, Agrawal O, Tenreiro Machado J, editors, Advances in Fractional Calculus, The Netherlands: Springer: 2007, 549p.

[4] Baleanu D, Diethelm K, Scalas E, Trujillo J, Fractional Calculus: Models and Numerical Methods. Singapore: World Scientific Publishing, 2012, 400p.

[5] Levine H. and Ben-Jacob E, Physical schemata underlying biological pattern formation-examples, issues and strategies. Physical Biology, 1, 14-22, 2004.

[6] Adler J., Chemotaxis in bacteria, Science 153, 225–234, 1971.

[7] Golding I., Kozlovsky Y., Cohen I. and Ben-Jacob E, Studies of bacterial branching growth using reaction-diffusion models for colonial development, Physica A 260, 510–554, 1998.

[8] Matsushita M., Wakita, J. I. and Matsuyama T. (1995). Growth and morphological changes of bacteria colonies, Santa-Fe Institute Studies in the Sciences of Complexity, Addison-Wesley, 609–618, 1995.

[9] El-sayed A.M.A, Rida S.Z. and Arafa A.A.M., On the solutions of the generalized reaction- diffusion model for Bacterial Colony, Acta Applicandae mathematicae, 110,1501-1511, 2010.

[10] Salah A. and Hassan S. S. A., A general algorithm for solving the fractional reaction diffusion model arising in bacterial colony by Homotopy Analysis Transform Method, World Applied Sciences Journal, 21(5), 659-664, 2013.

[11] Haidong Q, Rahman M, Shabir A, Bilal Riaz, Ibrahim M and Saeed T, Investigation of fractional order bacteria dependent disease with the effects of different contact rates, Choas, Solitons & Fractals, 159, 112169, 2022.

[12] Yeolekar M.A, Khirsaiya S.R, Chauhan J. P and Yeolekar B.M., Fractinal order dynamics in bacterial disease propagation: Modeling and analysis, Fracnkin Open, 12, 100283, 2025.

[13] Podlubny I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Vol. 198. San Diego, London, and Toronto: Academic Press; 1999.

[14] Samko S.G, Kilbas A.A. and Marichev O. I., Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, USA, 1993.

[15] Miller K. S., and Ross B, An introduction to the fractional calculus and fractional differential equations. John Wiley and Sons, 1993.

[16] Abdel Rahim Mahgoub M. M., The New Integral Transform Mahgoub Transform, *Advances in Theoretical and Applied Mathematics*, 11, 391-398, 2016.

[17] Emimal, K.P. and Lydia, K, Mahgoub Transform Method for solving linear fractional differential equations, International Journal of Mathematics Trends and Technology, 58(4), 253-257, 2018.