Google Scholar Crossref doi scopus

Impact factor 6.2

Geoscience Journal

ISSN:1000-8527

Indexing:

- Scopus
- Google Scholar
- DOI, Zenodo
- Open Access

www.geoscience.ac

GEOSCIENCE JOURNAL ESTD. 2020 ISSN:1000-8527

Registered

Some rational type contractions on F-modular b-complete metric spaces and its application to Nonlinear Integral Equation

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Abstract

In this article, we propose a new generalized framework, termed an *F-modular b-complete metric space*, which naturally combines the features of modular metric spaces, b-metric spaces, and F-metric spaces. This structure allows a unified treatment of several distance-type notions that have appeared independently in the literature. Within this setting, we establish analogues of classical fixed point results, including Banach, Kannan, Chatterjea, and several rational-type contraction principles for self-mappings. The obtained results extend and refine many existing fixed point theorems in generalized metric spaces. To illustrate the applicability of the proposed framework, one of the derived fixed point theorems is used to establish the existence and uniqueness of a solution of a nonlinear integral equation.

Keywords: b-metric space, Modular metric space, F-modular space, fixed point, rational type contraction

*AMS classification 2020:*47H10, 46A80, 54H25

1 Introduction

Over the last few years, fixed point theory has evolved as one of the most important areas of research in mathematical analysis due to its wide range of applications. Banach's contraction principle plays a crucial role in solving nonlinear differential and integral equations. Since its introduction, numerous generalizations and extensions have been developed.

The concept of b-metric spaces, introduced by Bakhtin[2] and later formalized by Czerwik[8], generalizes the classical notion of metric spaces. Parallelly, modular metric spaces, initiated by Nakano[20] and further developed by Chistyakov[5, 6, 6], offer another powerful generalization. Since then numorous authors have studied modular metric spaces by using different contractive conditions in [3, 10, 16, 17, 26]. The notion of F-metric spaces introduced by Jleli and Samet [13] provides a unified framework encompassing many known metric-type structures.

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Das and Gupta [9] have extended the Banach contraction mapping by introducing the concept of rational expressions in contractive mappings. They have introduced the following rational type contractive map

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y) \tag{1}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ to prove fixed point theorems in a complete metric space. Later on, numorous authors have studied fixed point theorems in different modular spaces using rational type contractions [18, 19, 22].

Inspired by these developments, the present work introduces a new structure called an *F-modular b-metric space* with rational type contraction. This framework merges the essential characteristics of modular metric spaces, b-metric spaces, and F-metric spaces, thereby offering a broader and more flexible setting for fixed point investigations. Within this newly defined space, we establish several fixed point theorems corresponding to Banach-type, Kannan-type, Chatterjea-type, and various rational-type contractions. Our results not only generalize many well-known theorems but also highlight the effectiveness of the proposed structure in unifying different approaches in fixed point theory.

2 Foundational Concepts and Preliminaries

Definition 2.1. [2] Let S be a nonempty set. A function $\rho : S \times S \to [0, \infty)$ is called a *b-metric* if for all $x, y, z \in S$ and $b \geq 1$:

(1) $\rho(x, y) = 0$ if and only if $x = y$,

(2) $\rho(x, y) = \rho(y, x)$,

(3) $\rho(x, z) \leq b[\rho(x, y) + \rho(y, z)]$.

The pair (S, ρ) is called a b–metric space.

Let S be a nonempty set, $\lambda > 0$, $\omega : (0, \infty) \times S \times S \to [0, \infty]$. We write $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$, $x, y \in S$ so that $\omega = \{\omega_\lambda\}_{\lambda>0}$ for which $\omega_\lambda : S \times S \to [0, \infty]$.

Definition 2.2. [16] Let S be any arbitrary vector space. A function $\rho : S \to [0, \infty]$ is called a modular on S if for any arbitrary $x, y \in S$

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

If (iii) is replaced by $(iii)'\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, we say that ρ is a convex modular.

Definition 2.3. [10] Let S be a nonempty set and let $\beta : (0, \infty) \times S \times S \to [0, \infty)$ be a mapping. Then β is called a *modular b-metric* if for all $u, v, q \in S$ and $\mu_1, \mu_2 > 0$:

(1) $\beta_\mu(u, v) = 0$ for all $\mu > 0$ if and only if $u = v$,

(2) $\beta_\mu(u, v) = \beta_\mu(v, u)$,

(3) $\beta_{\mu_1+\mu_2}(u, v) \leq b[\beta_{\mu_1}(u, q) + \beta_{\mu_2}(q, v)]$.

The pair (S, β) is called a modular b-metric space.

Definition 2.4. [14] Let Π denote the family of all functions $\pi : (0, \infty) \to \mathbb{R}$ satisfying:

1. π is nondecreasing,
2. $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \pi(t_n) = -\infty$.

Such a function is called a *logarithmic-like function*.

Definition 2.5. [25] Let S be a nonempty set and let $P : S \times S \to [0, +\infty)$ be a given mapping. Suppose that there exists $(\pi, \lambda) \in \Pi \times [0, +\infty)$ such that for all $(u, v) \in S \times S$, the following conditions hold:

1. $P(u, v) = 0$ if and only if $u = v$;
2. $P(u, v) = P(v, u)$;
3. for every $N \in \mathbb{N}$, $N \geq 2$, and for every $\{u_i\}_{i=1}^N \subset S$ with $(u_1, u_N) = (u, v)$, we have

$$P(u, v) > 0 \implies \pi(P(u, v)) \leq \pi\left(\sum_{i=1}^{N-1} P(u_i, u_{i+1})\right) + \lambda.$$

Then P is called an *F-metric* on S, and the pair (S, P) is called an *F-metric space*.

Definition 2.6. [25] Let S be a nonempty set and let $\gamma_\mu : (0, +\infty) \times S^2 \to [0, +\infty)$ be a function. If there exists $(\pi, \lambda) \in \Pi \times [0, +\infty)$ such that for all $u, v \in S$, the following conditions are satisfied:

1. $\gamma_\mu(u, v) = 0$ if and only if $u = v$;
2. $\gamma_\mu(u, v) = \gamma_\mu(v, u)$;
3. for all $n \in \mathbb{N}$, $n > 2$, and for any $\{v_1, v_2, \ldots, v_n\} \subset S$ with $(v_1, v_n) = (u, v)$, we have

$$\gamma_\mu(u, v) > 0 \implies \pi(\gamma_\mu(u, v)) \leq \pi\left(\sum_{j=1}^{n-1} \gamma_{\mu_j}(v_j, v_{j+1})\right) + \lambda.$$

Then γ_μ is called a *modular F-metric* on S, and (S, γ_μ) is called a *modular F-metric space*.

Definition 2.7 ([21])**.** Let S be a nonempty set and let $b \geq 1$ be a real number. Let $(\pi, \lambda) \in \Pi \times [0, +\infty)$. A mapping $\rho : S \times S \to [0, +\infty)$ is called a *function weighted b–metric* (or *F–b–metric*) if for all $u, v \in S$, the following conditions hold:

1. $\rho(u, v) = 0$ if and only if $u = v$;
2. $\rho(u, v) = \rho(v, u)$;
3. if $\rho(u, v) > 0$, then
$$\pi(\rho(u, v)) \leq \pi\left(\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})\right) + \lambda,$$
for every finite sequence $\{a_1 = u, a_2, \ldots, a_N = v\} \subset S$ with $N \geq 2$.

The pair (S, ρ) is called a *function weighted b-metric space.*

Definition 2.8. [25] Let S be a nonempty set and let $\gamma_\mu : (0, \infty) \times S \times S \to [0, \infty)$ be a continuous mapping. If there exist $(\pi, \lambda) \in \Pi \times [0, \infty)$ and $b > 1$ such that for all $u, v \in S$:

1. $\gamma_\mu(u, v) = 0$ if and only if $u = v$,
2. $\gamma_\mu(u, v) = \gamma_\mu(v, u)$,
3. for any finite sequence $\{v_1 = u, \ldots, v_n = v\} \subset S$, $n > 2$,
$$\gamma_\mu(u, v) > 0 \Rightarrow \pi(\gamma_\mu(u, v)) \leq \pi\left(\sum_{j=1}^{n-1} b^j \gamma_{\mu_j}(v_j, v_{j+1})\right) + \lambda,$$
where $\mu = \sum_{j=1}^{n-1} \mu_j$.

Then (S, γ_μ) is called an *F-modular b-metric space.*

Example 2.1. Let $S = \mathbb{R}$ and define
$$\gamma_\mu(u, v) = \frac{|u - v|^\kappa}{\mu}, \qquad \kappa \geq 2.$$
Then (S, γ_μ) is an F-modular b-metric space.

Definition 2.9. [25] Let (S, γ_μ) be an F-modular b-metric space and $\{v_n\} \subset S$ be a sequence.

1. The sequence $\{v_n\}$ is said to be *F-modular b-convergent* to $v \in S$ if
$$\lim_{n\to\infty} P_\mu(v_n, v) = 0.$$
2. The sequence $\{v_n\}$ is said to be an *F-modular b-Cauchy sequence* if
$$\lim_{m,n\to\infty} \gamma_\mu(v_m, v_n) = 0.$$

3. The space (S, γ_μ) is said to be *F-modular b-complete* if every F-modular b-Cauchy sequence in S is F-modular b-convergent to some element of S.

Definition 2.10. [14] Let (S, γ_μ) be an F-modular b-metric space. A mapping $R : S \to S$ is called an *F-modular b-contraction* if there exists $k \in (0, 1)$ such that

$$\gamma_\mu(Ru, Rv) \leq k\gamma_\mu(u, v), \qquad \forall u, v \in S.$$

3 Fixed point thoerems in F-Modular b-complete metric space

Theorem 3.1. *[25] Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. If a mapping $R : S \to S$ is an F-modular b-contraction, that is, there exists a constant $k \in (0, 1)$ such that*

$$\gamma_\mu(Rv, Rw) \leq k\,\gamma_\mu(v, w), \qquad \forall\, v, w \in S, \tag{2}$$

with $bk < 1$, then the mapping R admits a unique fixed point in S.

Theorem 3.2. *Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. Assume that a mapping $R : S \to S$ satisfies*

$$\gamma_\mu(Rv, Rw) \leq k\,\frac{\gamma_\mu(v, w)}{1 + \gamma_\mu(v, w)}, \qquad \forall\, v, w \in S, \tag{3}$$

where $0 < k < 1$ and $b > 1$ such that $0 < bk < 1$. Then the mapping R admits a unique fixed point in S.

Proof. Let $a_0 \in S$ be arbitrary and define a sequence $\{a_n\}$ in S by

$$a_{n+1} = R(a_n), \qquad n \in \mathbb{N} \cup \{0\}. \tag{4}$$

Since R is F-modular b-contraction, using (3) with $v = a_{n-1}$ and $w = a_n$, we obtain

$$\gamma_\mu(a_n, a_{n+1}) = \gamma_\mu(Ra_{n-1}, Ra_n) \leq k\,\frac{\gamma_\mu(a_{n-1}, a_n)}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

Since

$$\frac{t}{1+t} \leq t \quad \text{for all } t \geq 0,$$

it follows that

$$\gamma_\mu(a_n, a_{n+1}) \leq k\,\gamma_\mu(a_{n-1}, a_n), \qquad \forall n \geq 1. \tag{5}$$

By induction, we obtain

$$\gamma_\mu(a_n, a_{n+1}) \leq k^n \gamma_\mu(a_0, a_1), \qquad \forall n \in \mathbb{N}. \tag{6}$$

Now we prove that $\{a_n\}$ is an F-modular b-Cauchy sequence. Let $m > n$. By property (P3) of Definition 3.1, we have

$$\gamma_\mu(a_n, a_m) \le \sum_{j=n}^{m-1} b^{j-n+1} \gamma_{\mu_j}(a_j, a_{j+1}),$$

where $\mu = \sum_{j=1}^{m-n} \mu_j$. Using (6) and choosing $\mu_1 = \mu_2 = \cdots = \mu_{m-n} = \mu/(m-n)$, we get

$$\gamma_\mu(a_n, a_m) \le \sum_{j=n}^{m-1} b^{j-n+1} k^j \gamma_{\mu/(m-n)}(a_0, a_1).$$

Hence,

$$\gamma_\mu(a_n, a_m) \le \left(bk^n + b^2 k^{n+1} + \cdots + b^{m-n} k^{m-1}\right) \gamma_{\mu/(m-n)}(a_0, a_1).$$

Since $bk < 1$, we obtain

$$\gamma_\mu(a_n, a_m) \le \frac{bk^n}{1 - bk} \gamma_{\mu/(m-n)}(a_0, a_1). \tag{7}$$

But $\lim_{n\to\infty} \frac{bk^n}{1 - bk} \gamma_{\mu/(m-n)}(a_0, a_1) = 0$. So we can find an $\varepsilon > 0$ such that

$$0 \le \frac{bk^n}{1 - bk} \gamma_{\mu/(m-n)}(a_0, a_1) < \varepsilon. \tag{8}$$

Since π is increasing, hence for all $\varepsilon > 0$ and $r > 0$, there exists a $s > 0$ such that $\pi(s) < \pi(r) < \pi(\varepsilon) - \lambda$ whenever $0 < s < r$.

Hence for all $m > n > n_0$ and from equation(7) and equation(8), we get

$$\begin{aligned}
\pi(\gamma_\mu(a_n, a_m)) &\le \pi\left(\sum_{j=n}^{m-1} b^j \gamma_\mu(a_j, a_{j+1})\right) \\
&\le \pi\left(\frac{bk^n}{1 - bk} \gamma_{\mu/(m-n)}(a_0, a_1)\right) < \pi(\varepsilon) - \lambda \\
\Rightarrow \pi(\gamma_\mu(a_n, a_m)) &\le \pi(\gamma_\mu(a_n, a_m)) + \lambda < \pi(\varepsilon)
\end{aligned}$$

which implies $\gamma_\mu(a_n, a_m) < \varepsilon$ for all $m > n > n_0$.Thus, $\{a_n\}$ is an F-modular b-Cauchy sequence.

Since (S, γ_μ) is F-modular b-complete, there exists $a \in S$ such that

$$\lim_{n\to\infty} \gamma_\mu(a_n, a) = 0.$$

Assume that $Ra \neq a$. Then $\gamma_\mu(Ra, a) > 0$. By property (P3) of Definition 3.1, we have

$$\pi(\gamma_\mu(Ra, a)) \le \pi\left(b\gamma_{\mu/2}(Ra, Ra_n) + b\gamma_{\mu/2}(Ra_n, a)\right) + \lambda.$$

Using (3) and letting $n \to \infty$, we obtain

$$\pi(\gamma_\mu(Ra, a)) = -\infty,$$

which is a contradiction. Hence $Ra = a$.

Next we claim that the fixed point is unique. Suppose that $a^{'}, \bar{a} \in S$ are two distinct fixed points of R. Then

$$\gamma_\mu(a^{'}, \bar{a}) = \gamma_\mu(Ra^{'}, R\bar{a}) \leq k\, \frac{\gamma_\mu(a^{'}, \bar{a})}{1 + \gamma_\mu(a^{'}, \bar{a})} < \gamma_\mu(a^{'}, \bar{a}),$$

which is impossible. Hence $a^{'} = \bar{a}$.

Therefore, R has a unique fixed point in S. □

Example 3.1. The condition $bk < 1$ in Theorem 4.2 is essential. For example, let $S = \mathbb{R}$ and define

$$\gamma_\mu(v, w) = \frac{|v - w|}{\mu}, \quad \mu > 0.$$

Then (S, γ_μ) is an Fmodular $b$metric space with $b = 2$. Moreover, (S, γ_μ) is Fmodular $b$complete.

Define a mapping $R : S \to S$ by

$$R(v) = v + 1.$$

For all $v, w \in S$, we have

$$\gamma_\mu(Rv, Rw) = \frac{|(v + 1) - (w + 1)|}{\mu} = \frac{|v - w|}{\mu} = \gamma_\mu(v, w).$$

Hence,

$$\gamma_\mu(Rv, Rw) \leq k \frac{\gamma_\mu(v, w)}{1 + \gamma_\mu(v, w)} \quad \text{holds for } k = 1.$$

However, $k = 1$ implies $bk = 2 \geq 1$, so the hypothesis $bk < 1$ of Theorem 4.2 is violated. Clearly, the mapping $R(v) = v + 1$ has no fixed point in S.

Thus, without the condition $bk < 1$, the conclusion of Theorem 3.2 fails.

Theorem 3.3. *[25] Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. Suppose that a continuous mapping $R : S \to S$ satisfies*

$$\gamma_\mu(Rv, Rw) \leq k\big[\gamma_\mu(v, Rv) + \gamma_\mu(w, Rw)\big], \qquad \forall\, v, w \in S, \tag{9}$$

where $0 \leq k < \frac{1}{2}$ and $b > 1$ such that $0 < bk < 1$. Then R admits a unique fixed point in S.

Theorem 3.4. *Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. Assume that a mapping $R : S \to S$ satisfies*

$$\gamma_\mu(Rv, Rw) \leq k\, \frac{\gamma_\mu(v, Rv)\, \gamma_\mu(w, Rw)}{1 + \gamma_\mu(v, w)}, \qquad \forall\, v, w \in S, \tag{10}$$

where $k > 0$ and $b > 1$ are such that $0 < bk < 1$. Then R admits a unique fixed point in S.

Proof. Let $a_0 \in S$ be arbitrary and define a sequence $\{a_n\}$ in S by

$$a_{n+1} = R(a_n), \qquad n \in \mathbb{N} \cup \{0\}. \tag{11}$$

Since R is F-modular b-contraction, applying (10) with $v = a_{n-1}$ and $w = a_n$, we obtain

$$\gamma_\mu(a_n, a_{n+1}) = \gamma_\mu(Ra_{n-1}, Ra_n) \le k\, \frac{\gamma_\mu(a_{n-1}, a_n)\, \gamma_\mu(a_n, a_{n+1})}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

If $\gamma_\mu(a_n, a_{n+1}) = 0$, then $a_n = a_{n+1}$ and hence a_n is a fixed point of R. Assume therefore that $\gamma_\mu(a_n, a_{n+1}) > 0$. Dividing both sides by $\gamma_\mu(a_n, a_{n+1})$, we get

$$1 \le k\, \frac{\gamma_\mu(a_{n-1}, a_n)}{1 + \gamma_\mu(a_{n-1}, a_n)} < k.$$

Since $bk < 1$, it follows that $k < 1$, which yields a contradiction. Hence,

$$\gamma_\mu(a_n, a_{n+1}) = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{12}$$

Thus,

$$a_n = a_{n+1} = R(a_n), \qquad \forall n,$$

and the sequence $\{a_n\}$ becomes eventually constant.

Next we claim the existence of a fixed point. From (12), there exists $a \in S$ such that

$$R(a) = a.$$

Hence, a is a fixed point of R.

Now we prove that the fixed point is unique. Suppose that $a^{'}, \bar{a} \in S$ are two fixed points of R with $a^{'} \neq \bar{a}$. Then

$$\gamma_\mu(a^{'}, \bar{a}) = \gamma_\mu(Ra^{'}, R\bar{a}) \le k\, \frac{\gamma_\mu(a^{'}, Ra^{'})\, \gamma_\mu(\bar{a}, R\bar{a})}{1 + \gamma_\mu(a^{'}, \bar{a})} = 0,$$

which implies $a^{'} = \bar{a}$, a contradiction. Therefore, the fixed point is unique.

Hence, R has a unique fixed point in S. □

Theorem 3.5. *Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. Assume that a mapping $R : S \to S$ satisfies*

$$\gamma_\mu(Rv, Rw) \le k\, \frac{\gamma_\mu(v, Rv) + \gamma_\mu(w, Rw)}{1 + \gamma_\mu(v, w)}, \qquad \forall\, v, w \in S, \tag{13}$$

where $0 < k < \frac{1}{2}$ and $b > 1$ such that $0 < bk < 1$. Then R admits a unique fixed point in S.

Proof. Let $a_0 \in S$ be arbitrary and define a sequence $\{a_n\}$ in S by

$$a_{n+1} = R(a_n), \qquad n \in \mathbb{N} \cup \{0\}. \tag{14}$$

Since R is F-modular b-contraction, using (13) with $v = a_{n-1}$ and $w = a_n$, we obtain

$$\gamma_\mu(a_n, a_{n+1}) = \gamma_\mu(Ra_{n-1}, Ra_n) \leq k \, \frac{\gamma_\mu(a_{n-1}, a_n) + \gamma_\mu(a_n, a_{n+1})}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

Since $1 + \gamma_\mu(a_{n-1}, a_n) \geq 1$, we have

$$\gamma_\mu(a_n, a_{n+1}) \leq k\big[\gamma_\mu(a_{n-1}, a_n) + \gamma_\mu(a_n, a_{n+1})\big].$$

Thus,

$$(1-k)\gamma_\mu(a_n, a_{n+1}) \leq k\gamma_\mu(a_{n-1}, a_n),$$

which implies

$$\gamma_\mu(a_n, a_{n+1}) \leq \frac{k}{1-k}\, \gamma_\mu(a_{n-1}, a_n). \tag{15}$$

Since $0 < k < \frac{1}{2}$, the constant

$$\beta = \frac{k}{1-k} \in (0, 1).$$

By induction, we obtain

$$\gamma_\mu(a_n, a_{n+1}) \leq \beta^n \gamma_\mu(a_0, a_1), \qquad \forall n \in \mathbb{N}. \tag{16}$$

Now we prove that $\{a_n\}$ is an F-modular b-Cauchy sequence. Let $m > n$. By property (P3) of Definition 3.1, we have

$$\gamma_\mu(a_n, a_m) \leq \sum_{j=n}^{m-1} b^{j-n+1} \gamma_{\mu_j}(a_j, a_{j+1}),$$

where $\mu = \sum_{j=1}^{m-n} \mu_j$. Using (16) and choosing $\mu_1 = \mu_2 = \cdots = \mu_{m-n} = \mu/(m-n)$, we get

$$\gamma_\mu(a_n, a_m) \leq \sum_{j=n}^{m-1} b^{j-n+1} \beta^j \gamma_{\mu/(m-n)}(a_0, a_1).$$

Hence,

$$\gamma_\mu(a_n, a_m) \leq \left(b\beta^n + b^2\beta^{n+1} + \cdots + b^{m-n}\beta^{m-1}\right) \gamma_{\mu/(m-n)}(a_0, a_1).$$

Since $b\beta < 1$ (because $bk < 1$), we obtain

$$\gamma_\mu(a_n, a_m) \leq \frac{b\beta^n}{1 - b\beta}\, \gamma_{\mu/(m-n)}(a_0, a_1).$$

Letting $n \to \infty$, we conclude that

$$\lim_{n,m\to\infty} \gamma_\mu(a_n, a_m) = 0.$$

Thus, $\{a_n\}$ is an F-modular b-Cauchy sequence.

Since (S, γ_μ) is F-modular b-complete, there exists $a \in S$ such that

$$\lim_{n\to\infty} \gamma_\mu(a_n, a) = 0.$$

Using (13) with $v = a_n$ and $w = a$, we have

$$\gamma_\mu(a_{n+1}, Ra) \leq k\,\frac{\gamma_\mu(a_n, a_{n+1}) + \gamma_\mu(a, Ra)}{1 + \gamma_\mu(a_n, a)}.$$

Letting $n \to \infty$ and using (16), we obtain

$$\gamma_\mu(a, Ra) \leq k\gamma_\mu(a, Ra).$$

Since $0 < k < 1$, this implies $\gamma_\mu(a, Ra) = 0$, and hence

$$Ra = a.$$

Next we will prove the uniqueness of the fixed point. Suppose that $a^{'}, \bar{a} \in S$ are two fixed points of R with $a^{'} \neq \bar{a}$. Then

$$\gamma_\mu(a^{'}, \bar{a}) = \gamma_\mu(Ra^{'}, R\bar{a}) \leq k\,\frac{\gamma_\mu(a^{'}, Ra^{'}) + \gamma_\mu(\bar{a}, R\bar{a})}{1 + \gamma_\mu(a^{'}, \bar{a})} = 0,$$

which is a contradiction. Hence $a^{'} = \bar{a}$.

Therefore, R has a unique fixed point in S. □

Theorem 3.6. *Let S be a nonempty set endowed with an F-modular b-metric γ_μ such that (S, γ_μ) is F-modular b-complete. Assume that a mapping $R : S \to S$ satisfies*

$$\gamma_\mu(Rv, Rw) \leq k\,\frac{\gamma_\mu(v, Rw) + \gamma_\mu(w, Rv)}{1 + \gamma_\mu(v, w)}, \qquad \forall\, v, w \in S, \tag{17}$$

where $0 < k < \frac{1}{2}$ and $b > 1$ such that $0 < bk < 1$. Then R admits a unique fixed point in S.

Proof. Let $a_0 \in S$ be arbitrary and define a sequence $\{a_n\}$ in S by

$$a_{n+1} = R(a_n), \qquad n \in \mathbb{N} \cup \{0\}. \tag{18}$$

Since R is F-modular b-contraction, using (17) with $v = a_{n-1}$ and $w = a_n$, we obtain

$$\gamma_\mu(a_n, a_{n+1}) = \gamma_\mu(Ra_{n-1}, Ra_n) \leq k\,\frac{\gamma_\mu(a_{n-1}, a_{n+1}) + \gamma_\mu(a_n, a_n)}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

Since $\gamma_\mu(a_n, a_n) = 0$, this reduces to

$$\gamma_\mu(a_n, a_{n+1}) \leq k\,\frac{\gamma_\mu(a_{n-1}, a_{n+1})}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

Using the b-triangle inequality (property (P3)), we have

$$\gamma_\mu(a_{n-1}, a_{n+1}) \leq b\big[\gamma_\mu(a_{n-1}, a_n) + \gamma_\mu(a_n, a_{n+1})\big].$$

Hence,

$$\gamma_\mu(a_n, a_{n+1}) \leq k\, \frac{b\big[\gamma_\mu(a_{n-1}, a_n) + \gamma_\mu(a_n, a_{n+1})\big]}{1 + \gamma_\mu(a_{n-1}, a_n)}.$$

Since $1 + \gamma_\mu(a_{n-1}, a_n) \geq 1$, we obtain

$$\gamma_\mu(a_n, a_{n+1}) \leq kb\big[\gamma_\mu(a_{n-1}, a_n) + \gamma_\mu(a_n, a_{n+1})\big].$$

Thus,

$$(1 - kb)\gamma_\mu(a_n, a_{n+1}) \leq kb\, \gamma_\mu(a_{n-1}, a_n),$$

which implies

$$\gamma_\mu(a_n, a_{n+1}) \leq \frac{kb}{1 - kb}\, \gamma_\mu(a_{n-1}, a_n). \tag{19}$$

Since $0 < k < \frac{1}{2}$ and $bk < 1$, the constant

$$\delta = \frac{kb}{1 - kb} \in (0, 1).$$

By induction, we obtain

$$\gamma_\mu(a_n, a_{n+1}) \leq \delta^n \gamma_\mu(a_0, a_1), \qquad \forall n \in \mathbb{N}. \tag{20}$$

Now we claim that $\{a_n\}$ is an F-modular b-Cauchy sequence. Let $m > n$. By property (P3) of Definition 3.1, we have

$$\gamma_\mu(a_n, a_m) \leq \sum_{j=n}^{m-1} b^{j-n+1} \gamma_{\mu_j}(a_j, a_{j+1}),$$

where $\mu = \sum_{j=1}^{m-n} \mu_j$. Using (20) and choosing $\mu_1 = \mu_2 = \cdots = \mu_{m-n} = \mu/(m-n)$, we get

$$\gamma_\mu(a_n, a_m) \leq \sum_{j=n}^{m-1} b^{j-n+1} \delta^j \gamma_{\mu/(m-n)}(a_0, a_1).$$

Hence,

$$\gamma_\mu(a_n, a_m) \leq \left(b\delta^n + b^2\delta^{n+1} + \cdots + b^{m-n}\delta^{m-1}\right) \gamma_{\mu/(m-n)}(a_0, a_1).$$

Since $b\delta < 1$, we obtain

$$\gamma_\mu(a_n, a_m) \leq \frac{b\gamma^n}{1 - b\gamma}\, \gamma_{\mu/(m-n)}(a_0, a_1).$$

Letting $n \to \infty$, we conclude that

$$\lim_{n,m\to\infty} \gamma_\mu(a_n, a_m) = 0.$$

Thus, $\{a_n\}$ is an F-modular b-Cauchy sequence.

Since (S, γ_μ) is F-modular b-complete, there exists $a \in S$ such that

$$\lim_{n\to\infty} \gamma_\mu(a_n, a) = 0.$$

Using (17) with $v = a_n$ and $w = a$, we have

$$\gamma_\mu(a_{n+1}, Ra) \leq k\,\frac{\gamma_\mu(a_n, Ra) + \gamma_\mu(a, a_{n+1})}{1 + \gamma_\mu(a_n, a)}.$$

Letting $n \to \infty$, we obtain

$$\gamma_\mu(a, Ra) \leq k\gamma_\mu(a, Ra).$$

Since $0 < k < 1$, this implies $\gamma_\mu(a, Ra) = 0$, and hence

$$Ra = a.$$

Now we prove the uniqueness of the fixed point. Suppose that $a^{'}, \bar{a} \in S$ are two fixed points of R with $a^{'} \neq \bar{a}$. Then

$$\gamma_\mu(a^{'}, \bar{a}) = \gamma_\mu(Ra^{'}, R\bar{a}) \leq k\,\frac{\gamma_\mu(a^{'}, R\bar{a}) + \gamma_\mu(\bar{a}, Ra^{'})}{1 + \gamma_\mu(a^{'}, \bar{a})} = 0,$$

which is a contradiction. Hence $a^{'} = \bar{a}$.

Therefore, R has a unique fixed point in S. □

4 Application to Nonlinear Integral Equation

In this section, we demonstrate the usefulness of the established fixed point results by applying Theorem 3.2 to ensure the existence and uniqueness of solutions for a class of nonlinear integral equations.

Consider the nonlinear integral equation

$$p(t) = \int_0^a K(t, s, p(s))\, ds, \quad t \in [0, a], \tag{21}$$

where $a > 0$ is a fixed real number and $K : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $S = C([0, a], \mathbb{R})$ denote the space of all real-valued continuous functions on $[0, a]$. Define a function $\gamma_\mu : S \times S \to [0, \infty)$ by

$$\gamma_\mu(x, y) = \frac{\sup\limits_{t\in[0,a]} |x(t) - y(t)|}{\mu}, \quad \mu > 0. \tag{22}$$

It is well known that (S, γ_μ) forms an F-modular b-metric space, and moreover, it is F-modular b-complete.

Define an operator $R : S \to S$ associated with the integral equation by

$$(Rp)(t) = \int_0^a K(t, s, p(s))\, ds, \quad t \in [0, a]. \tag{23}$$

Assume that there exists a constant $L > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \le L|u - v| \tag{24}$$

for all $t, s \in [0, a]$ and for all $u, v \in \mathbb{R}$, with the additional condition

$$La < 1.$$

Then, for any $p, q \in S$, we obtain

$$\begin{aligned}
\gamma_\mu(Rp, Rq) &= \frac{1}{\mu} \sup_{t\in[0,a]} \left| \int_0^a \big(K(t, s, p(s)) - K(t, s, q(s))\big)\, ds \right| \\
&\le \frac{1}{\mu} \sup_{t\in[0,a]} \int_0^a L|p(s) - p(s)|\, ds \\
&\le La\, \gamma_\mu(p, q).
\end{aligned}$$

Since $La < 1$, the operator R satisfies the rational-type contractive condition required in Theorem 3.2. Consequently, R admits a unique fixed point $x^* \in S$.

Therefore, the nonlinear integral equation admits a unique continuous solution on the interval $[0, a]$.

5 Conclusion

In this work, a new class of generalized distance spaces, referred to as F-modular b-metric spaces, has been introduced by combining the structural features of modular metrics, b-metrics, and F-metrics. This unified framework provides a flexible setting for the analysis of fixed point problems and extends several existing metric-type structures studied in the literature.

Within this setting, a variety of fixed point theorems have been established for self-mappings satisfying different contractive conditions, including Banach-type, Kannan-type, Chatterjea-type, and rational-type contractions. The obtained results generalize and improve many well-known fixed point theorems and demonstrate that the proposed structure is sufficiently rich to accommodate a wide class of nonlinear mappings.

To illustrate the applicability of the theoretical results, the developed fixed point theorems were applied to study the existence and uniqueness of solutions of nonlinear problems. In particular, one application was presented for a nonlinear integral equation. This application confirms that the F-modular b-metric framework is not only of theoretical interest but also serves as an effective tool for investigating nonlinear equations arising in analysis.

The results of this paper open several directions for future research. Possible extensions include the study of multivalued mappings, common fixed point results, and applications to integro-differential equations and fractional differential equations within the setting of F-modular b-metric spaces. Such investigations may further enhance the scope and applicability of the proposed framework.

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